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A Nebeský-Type Characterization for Relative Maximum Genus

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This paper concerns the maximum genus orientable surface upon which a given graph cellularly embeds. Classical theorems of Xuong and Nebeský give exact values for the maximum genus. The former is suited to constructing embeddings while the latter is suited to forbidding embeddings of larger genus. However, using either theorem alone requires an exhaustive search to establish the exact value. Herein we examine relative embeddings of graphs, where certain facial cycles and their orientations have been prescribed. The relative graph analogue of Xuong's theorem is known. In this paper we establish the relative graph analogue of Nebeský's theorem.

1. INTRODUCTION AND PRELIMINARIES

This paper concerns embedding graphs into orientable manifolds. From a theorem of Brahana [2], these surfaces are homeomorphic to a sphere with g handles attached; this g is called the *genus* of the surface. By convention we consider only embeddings of graphs where the complement

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in the surface of the embedded graph consists of open 2-cells (*cellular* embeddings). Among all such cellular embeddings of a fixed graph G, it is interesting to examine those into surfaces of minimum or maximum genus. By a theorem of Duke [3], G embeds into all surfaces whose genus lies between these two extreme values, so they determine the spectrum of the genera of surfaces on which G embeds. The minimum genus of a surface admitting a given graph is a difficult parameter. Here we focus primarily on the maximum genus of G, denoted $\gamma_M(G)$.

Let G be a graph with $|V| = |V(G)|$ vertices and $|E| = |E(G)|$ edges. Suppose that G is cellularly embedded in a surface of genus g with $|F|$ faces. Then the Euler–Poincaré formula states that $|V| - |E| + |F| = 2 - 2g$. Thus an embedding of maximum genus corresponds to an embedding with the minimum number of faces. This formula also implies that the parity of the number of faces is determined by the parity of $|V| - |E|$. Specifically, define the *Betti number* of G as $\beta(G) = |C| - |V| + |E|$, where C is the set of components of G. Then $\beta(G)$ differs in parity from $|F|$ for every embedding of G.

There are two main theorems which help to determine the maximum genus of a graph. These are due to Xuong [11] and Nebeský [5], respectively. (These results were significant advances upon the basic work of Nordhaus, *et al.* [6, 7].) The easy half of Xuong's theorem states:

THEOREM 1. *Suppose that G is a connected graph. Let T be a spanning tree of G, and let $\omega(T)$ be the number of components of $G - T$ with an odd number of edges. Then there exists an embedding of G into an orientable surface with $1 + \omega(T)$ faces.*

This theorem is important because it gives a conveniently determined lower bound on the maximum genus of a graph. Namely, all one must do is establish a spanning tree with a small number of odd cardinality co-components. The theorem then asserts the existence of an embedding with a small number of faces and, hence, with a large genus. The hard half of Xuong's theorem states that this is the only obstruction to a large genus embedding. Namely, to find the maximum genus of G one needs only to examine the minimum value of $1 + \omega(T)$ over all spanning trees T of G. However, due to the large number of spanning trees this may be a difficult (although polynomial [4]) task.

The theorem of Nebeský works in the other direction; the easy half states:

THEOREM 2. *Suppose that G is a connected graph. Let A be a subset of edges, let $c(A)$ be the number of components of $G - A$, and let $o(A)$ be the number of components of odd Betti number in $G - A$. Then any embedding of G has at least $c(A) + o(A) - |A|$ faces.*

This theorem is important because it gives a conveniently determined upper bound on the maximum genus of a graph. Namely, all one must do is to find a set $|A|$ of edges with a large value of $c(A)+o(A)-|A|$. The theorem then asserts that every embedding has at least this many faces and, hence, bounds above the maximum genus. The hard half of Nebeský's theorem states that these obstructions are the only ones to a large genus embedding. Namely, to find the maximum genus of G one needs only to examine the maximum value of $c(A)+o(A)-|A|$ over all subsets $A \subseteq E$ of edges of G. However, due to the large number of subsets this may be a difficult task.

The two theorems are especially powerful in concert. Specifically, using Xuong's theorem one can easily demonstrate the desired embedding, and using Nebeský's theorem assert that it is of maximum genus. Using these theorems in concert avoids the use of the extensive (and exhaustive!) searches needed when either one is used alone.

A rich area of research in recent years concerns the study of embeddings subject to particular restrictions. For example, Širáň and Škoviera [9] investigate the maximum genus for graph embeddings where the orientation-preserving and orientation-reversing cycles are prescribed.

We are particularly interested in embeddings where certain face boundaries and their orientations are prescribed. A motivating factor being the study of graph embeddings which are built from smaller pieces—one could imagine the prescribed faces containing other graph embeddings. Bonnington [1] gives a "Xuong-like" characterization for the maximum genus of such embeddings. That is, he gives a formula which leads to a method for constructing an embedding of large genus and, with an appropriate exhaustive search, gives the maximum such genus.

Our purpose here is to complement Bonnington's result by providing a "Nebeský-like" formula for the maximum genus of graph embeddings with some prescribed face boundaries. In fact, our main result, together with Bonnington's result, provides a "max–min" type characterization of a graph invariant, and for many such invariants polynomial-time algorithms are known. However, this is beyond the scope of the paper.

Let us focus more closely on graph embeddings in which certain face boundaries are prescribed. We first note that no edge can appear more than twice in prescribed face boundaries, for that is impossible in any embedding. Next, when an edge appears exactly twice in prescribed face boundaries we can delete that edge and merge the prescribed faces (keeping track of any handles which may be formed from such merges). Finally, when an edge appears in no face boundary, we can replace it with two edges in parallel and add a prescribed face boundary containing these two edges. These length-two face boundaries are called *fat edges*. By merging faces and replacing edges with fat edges we can assume that the prescribed face boundaries contain every edge exactly once.

Shortly, we present a more formal definition of a relative graph and introduce related concepts. To do this, we make extensive use of permutations. Let us adopt the convention that the composition of permutations is to be read from the right to the left. That is to say, if P and Q are two permutations on a set containing an element x, then $(QP)(x) = Q(P(x))$. We write id for the identity permutation.

We define a *relative graph*, G, to be an ordered triple (M, Π, Q), where M is a finite nonempty set, Π is a partition of M, and Q is a permutation of M. The members of M are *corners* (of G), the cells of Π are *vertices* (of G), and the orbits of Q are *inner faces* (of G or Q). This terminology allows one to say that a vertex *contains* its corners. Relative graphs in this form were introduced by Stahl [10] who called them permutation–partition pairs. Other authors have introduced equivalent formulations.

Associated with every relative graph (M, Π, Q) is a directed graph (digraph) H formed in the following way. The vertex set of H is Π; that is, each vertex of H is identified with a cell of the partition Π. Each corner u creates an arc in H joining the cell containing u to the cell containing $Q(u)$. We say that H is the digraph *underlying* (M, Π, Q). We note that there is a 1:1 correspondence between the arcs (edges) of H and the corners of (M, Π, Q).

EXAMPLE 1. Consider the relative graph $G = (M, \Pi, Q)$, where $M = \{1, 2, 3, 4, 5, 6\}$, $\Pi = \{\{1, 4\}, \{3, 5\}, \{2, 6\}\}$, and $Q = (1, 2, 3)(4, 5, 6)$. Figure 1 illustrates the digraph underlying G. Furthermore, the shaded regions identify the inner faces of G. We often regard such a figure as a "drawing" of the relative graph.

A relative graph is said to be *connected* if its underlying digraph is connected in the usual sense. In fact, we often ascribe properties of the

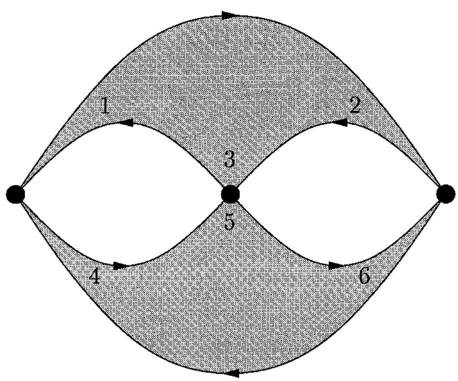

Figure 1

underlying digraph to the relative graph, and vice versa, without further explanation. For example, the components of a relative graph are loosely defined as relative graphs associated with the components of the underlying digraph.

An *embedding* of a relative graph G is a permutation P of the corners of G such that orbits of P coincide with the cells of Π (that is, the vertices of G). The orbits of QP are the *outer faces* of the embedding, and of the relative graph.

EXAMPLE 2. The anticlockwise cyclic ordering of the corners in each vertex of the relative graph illustrated in Fig. 1 lead to an embedding of the relative graph. Indeed, $P=(1,4)(3,5)(2,6)$ is the embedding and has outer faces $(1,5)$, $(2,4)$, and $(3,6)$. These outer faces are illustrated with dashed lines in Fig. 2. Also represented (with dotted lines) is the permutation P of the corners. We observe that the arrows on the outer faces and the permutation agree, while the arrows on the outer faces and the inner faces disagree.

From Example 2 one can easily see how embeddings of a relative graph correspond to oriented 2-cell embeddings of the underlying digraph; each face of the oriented 2-cell embedding is either an oriented "inner" face or an oriented "outer" face.

We shall be interested in embeddings of a relative graph G with the smallest number of outer faces, that is, in finding the minimum number of orbits of the permutation QP over all embeddings P of G. The reason for this is that, in the special case where all inner faces of G have length two (that is, they correspond to fat edges), the embedding of G with the minimum number of outer faces corresponds to the maximum genus embedding

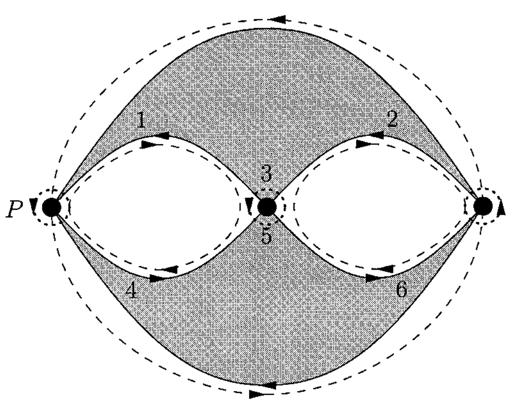

Figure 2

of the graph obtained from G by collapsing these fat edges to regular edges. Hence, as a corollary to our main theorem, one obtains the characterization of maximum genus due to Nebeský [5].

Let G be a connected relative graph with p vertices, q corners, and r inner faces. The *Betti number* $\beta(G)$ of G is defined as $\beta(G)=q-p+1-r$. There is a connection between the parities of the Betti number and the number of outer faces in any embedding of G: Consider an arbitrary 2-cell embedding of underlying digraph H in a surface of genus g with the r inner faces as some of the faces of this 2-cell embedding. If s denotes the number of outer faces, Euler's formula implies that $2-2g=p-q+(r+s)=s+1-\beta(G)$. Thus, if $\beta(G)$ is odd then the number of outer faces in *any* embedding of H (and hence G) is even and vice versa.

Let A be a permutation of a finite nonempty set. As it is well known, A has a unique decomposition into cyclic factors (up to the order of the factors). Denote by n_i the number of cycles of length i in this decomposition. Then the *weight* $|A|$ of A, is defined by $|A|=\sum_{i\geqslant 1}(n_i-1)$. Evidently, $|A|$ is the smallest number m for which the permutation A can be expressed as a product of m transpositions.

Our main result is:

THEOREM 3. *Let $G=(M, \Pi, Q)$ be a connected relative graph, and let A be an arbitrary permutation of M. Let $c_G(A)$ denote the number of components of (M, Π, AQ), and let $o_G(A)$ denote the number of components of (M, Π, AQ) that have an odd Betti number (briefly,* odd components*). Then the minimum number of outer faces in any embedding of G is*

$$\max\{c_G(A)+o_G(A)-|A|-1 : A \text{ is a permutation of } M\}+1.$$

In general, we denote by G_A the relative graph (M, Π, AQ) and by $y_G(A)$ the value for $c_G(A)+o_G(A)-|A|-1$. Furthermore, we let $y_G=\max y_G(A)$, where the maximum is taken over all permutations A of corners of G. Note that $y_G\geqslant 0$ for every relative graph G, since $y_G(\mathrm{id})\geqslant 0$.

Again, note that any specific instance of a subset A gives a lower bound on y_G and, hence, an upper bound on the maximum genus. Bonnington's generalization of Xuong's lower bound can be used to establish equality.

We pause at this point to explain the idea behind both Nebeský's theorem and our generalization. We begin with the graphical case. Suppose that we have an embedding of G with $|F|$ faces. Delete the edges in A one at a time. Each edge deletion increases the number of faces by at most one. Hence, the resulting embedding of $G-A$ has at most $|F|+|A|$ faces. But every component with even Betti number has at least one face, and every component with odd Betti number has at least two. This implies that the number of faces is at least $c(A)+o(A)$. Hence, $|F|+|A| \geqslant c(A)+o(A)$ which leads to the desired inequality (1 is subtracted and then added in because it makes y additive across components.) This gives the easy half of Nebeský's theorem. The hard half comes in finding an appropriate subset A of edges which reverses the above process in a maximum genus embedding.

The idea behind our generalization is similar. Consider a relative graph G embedded with $|F|$ outer faces. Now we change the embedded relative graph not by edge deletion but by multiplication by a transposition. Each such multiplication increases the number of outer faces by at most one. Hence one can construct an embedding of G_A with at most $|F|+|A|$ faces. As before this implies that $|F|+|A| \geqslant c_G(A)+o_G(A)$ which gives the desired inequality. The hard half of our main result is in establishing the existence of a permutation A which reverses the above process. This is significantly harder than in the graphical case and leads to the rather technical decomposition theorem of Section 4.

This paper is organized as follows. In Section 2 we give some basic facts about the function y_G. These facts lead to a rigorous proof of the "easy half" of our main theorem. Section 3 contains some auxilary results on permutations acting on set systems. Section 4 uses these results to obtain a Decomposition Theorem. Section 5 then completes the proof of our main result.

2. EXTREMAL AND CRITICAL PERMUTATIONS

In this section we prove a number of auxiliary results related to the function y_G. The proofs are easy and many details are therefore omitted.

LEMMA 1. *Let a and b be two distinct corners of a relative graph G and suppose that G has an embedding with $|F|$ outer faces. If the corners a and b appear in a single outer face (two different outer faces), then $G_{(a,b)}$ has an embedding with $|F|+1$ outer faces ($|F|-1$ outer faces).*

Proof. Let $G=(M, \Pi, Q)$. We know that $|F|$ is the number of orbits of the permutation QP for some rotation P. It is easy to show that the number of orbits of the permutation $(a, b)QP$ is then $|F| \pm 1$, depending on whether or not a and b belong to a single orbit of QP. ∎

For a relative graph $G=(M, \Pi, Q)$, Fig. 3 illustrates the effect that postmultiplying Q by a transposition has on the drawing of G.

A permutation A of corners of G is said to be *extremal* (*for* G) if $y_G = y_G(A)$.

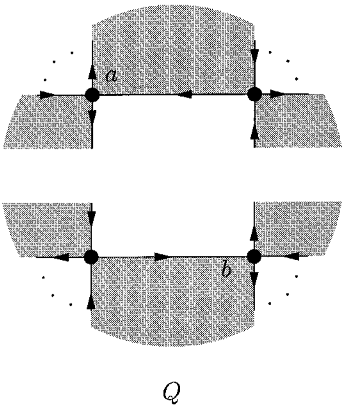

Figure 3

LEMMA 2. *Let A be an extremal permutation for a relative graph G. Then for every component H of G_A we have $y_H = 0$ or $y_H = 1$.*

Proof. Let H be an arbitrary component of G_A and let B be a permutation of the corners of H (formally, we assume that B is a permutation of all corners of G, fixing every corner that is not in H). Then we have

$$c_G(BA) = c_{G_A}(B) = c_G(A) - 1 + c_H(B),$$
$$o_G(BA) = o_{G_A}(B) \geqslant o_G(A) - 1 + o_H(B).$$

Since $|BA| \leqslant |B| + |A|$, then the above inequalities readily imply that

$$y_G \geqslant y_G(BA) \geqslant y_G(A) + y_H(B) - 1 = y_G + y_H(B) - 1.$$

Consequently, $y_H(B) \leqslant 1$ for every B; that is, $y_H \leqslant 1$. ∎

EXAMPLE 3. Consider the relative graph G of Example 3, and let A be the permutation $(1, 5)(3, 6)(2, 4)$. Then Fig. 4 gives a drawing of the relative graph G_A. While the 3 components of G_A are isomorphic, G_A has

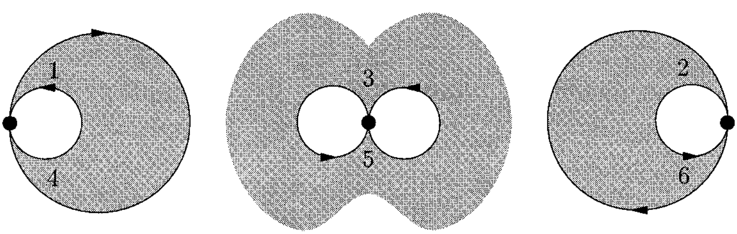

Figure 4

been illustrated in this way to establish its relationship with the drawing of G in Fig. 1. We observe that each component has Betti number 1. Thus $y_G(A) = 3+3-3-1 = 2$, and A is clearly an extremal permutation for G.

A connected relative graph G is said to be *prime* if the only extremal permutation for G is trivial. It is easy to see that a prime relative graph G must contain an inner face with more than one corner and that $y_G = 1$.

Let G be an arbitrary relative graph. A permutation A of corners of G is said to be *critical for* G if A is extremal and maximizes $c_G(A)$.

LEMMA 3. *Let A be a critical permutation for G. Then every component H of G_A with $y_H = 1$ is prime.*

Proof. Let B be an extremal permutation for the component H of G_A with $y_H = 1$. Similarly as in the preceding proof, we have the inequality $y_G \geqslant y_G(BA) \geqslant y_G + y_H(B) - 1$. Now, however, we know that $y_H(B) = y_H = 1$, and so BA is an extremal permutation for G. If $c_H(B) > 1$, then the extremal permutation BA would yield more components than A (that is, $c_G(BA)$ would be larger than $c_G(A)$), contrary to the fact that A is critical for G. Thus, $c_H(B) = 1$. But then, $o_H(B) \leqslant 1$, and

$$1 = y_H(B) = c_H(B) + o_H(B) - |B| - 1 \leqslant 1 - |B|.$$

This shows that $|B| \leqslant 0$, that is, B is trivial, and therefore H is prime. ∎

If a is a corner of a relative graph $G = (M, \Pi, Q)$ then we denote by $\Pi - a$ the partition obtained from Π by removing a from the cell in which it is a member. Further, we denote by $G - a$ the relative graph

$$(M - a, \Pi - a, (a, Q(a))Q).$$

(Note that $(a, Q(a))Q$ fixes a and we consider it as a permutation of $M - a$.) Figures 5a and b illustrate the implied effect on the drawing of G when Q permutes a, Q fixes a, respectively. When Q fixes a, there is an inner face of length 1, and hence, we say that a constitutes a *loop-face*.

LEMMA 4. *Every vertex of a prime graph G contains at least two corners. Moreover, if a constitutes a loop-face in G, then the relative graph $G - a$ is also prime.*

Proof. Assume that a vertex v of G contains a single corner a. The primeness of G implies that the corner $Q(a)$ is distinct from a. For $A = (a, Q(a))$, the relative graph G_A is disconnected and $1 = y_G = y_G(A)$, contrary to the primeness of G. Therefore, every vertex of G contains at least two corners.

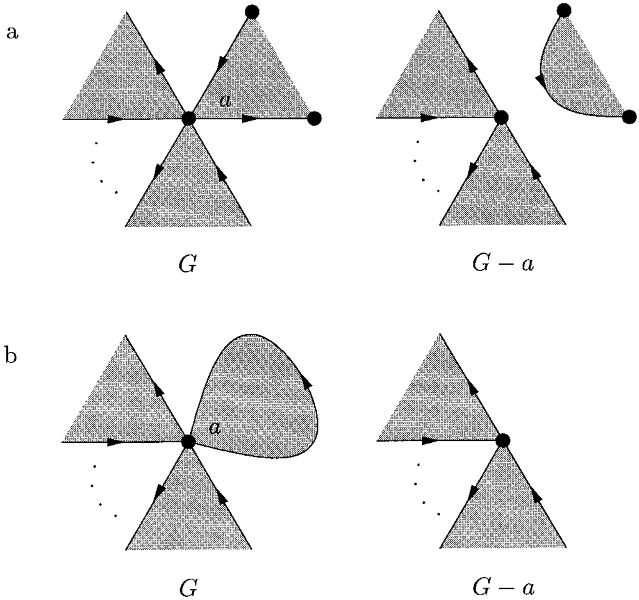

Figure 5

Now suppose the corner a constitutes a loop-face. Since G (being prime) has an odd Betti number, so has the relative graph $G-a$. It follows that $y_{G-a}(\mathrm{id})=1$, and so $y_{G-a}\geqslant 1$. Let B be an extremal permutation for $G-a$. Observe that adding the loop-face comprising the corner a back to G does not change the number of (odd) components; that is, $c_{G-a}(B)=c_G(B)$ and $o_{G-a}(B)=o_G(B)$. Consequently, $1\leqslant y_{G-a}=y_{G-a}(B)=y_G(B)\leqslant 1$, which shows that B is also extremal for G. However, G is prime and, therefore, B must be the identity permutation. This in turn shows that $G-a$ also is prime. ▮

Our last result in this section proves the "easy half" of Theorem 3.

PROPOSITION 1. *The minimum number of outer faces in any oriented embedding of a connected relative graph G is at least y_G+1.*

Proof. Consider an arbitrary embedding of G with, say, $|F|$ outer faces. Let A be an arbitrary permutation of the set of corners of G. Applying repeatedly Lemma 1, we see that G_A has an embedding with at most $|F|+|A|$ outer faces. Now, every odd component of G_A embeds with at least two outer faces, and every other component embeds with at least one outer face. It follows that $c_G(A)+o_G(A)\leqslant|F|+|A|$ and, hence,

$|F| \geqslant y_G(A)+1$. The last inequality is valid for every A and every $|F|$, thereby proving our lemma. ∎

3. N-SPACES

Let $\mathscr{S}$ be a finite collection of pairwise disjoint nonempty sets; we shall briefly refer to $\mathscr{S}$ as a *set system*. Denote by $\bigcup \mathscr{S}$ the union of all sets appearing in $\mathscr{S}$. Let A be an arbitrary permutation of the set $\bigcup \mathscr{S}$ and let $\mathscr{U}$ be a non-empty subset of $\mathscr{S}$. We define the *induced* permutation $A_{\mathscr{U}}$ of the set $\bigcup \mathscr{U}$ as follows: Let $x \in \bigcup \mathscr{U}$ and let l_x be the smallest positive integer l for which $A^l(x) \in \bigcup \mathscr{U}$. Now, $A_{\mathscr{U}}(x) = A^{l_x}(x)$.

In the application, we will begin with a relative graph G on the set of corners $\bigcup \mathscr{S}$. We will modify G to a relative graph $G_{A^{-1}}$. The corners in a component of $G_{A^{-1}}$ form one set $S \in \mathscr{S}$. In the current setting we find it easier to consider just the sets and the permutation A. Our first observation is on comparing weights of induced permutations.

Lemma 5. *Let $\mathscr{S}$ be a set system and let A be a permutation of $\bigcup \mathscr{S}$. Let $\mathscr{U}$ and $\mathscr{V}$ be subsets of $\mathscr{S}$ such that $\mathscr{U} \cap \mathscr{V} \neq \varnothing$. Then,*

$$|A_{\mathscr{U}}| + |A_{\mathscr{V}}| \leqslant |A_{\mathscr{U} \cup \mathscr{V}}| + |A_{\mathscr{U} \cap \mathscr{V}}|.$$

Proof. Without loss of generality we may assume that $\mathscr{U} \cup \mathscr{V} = \mathscr{S}$ and therefore $A_{\mathscr{U} \cup \mathscr{V}} = A$. Let $\mathscr{W} = \mathscr{U} \cap \mathscr{V}$. Consider an orbit $C = (x_1, x_2, ..., x_k)$ of A, with weight $|C| = k-1 > 0$. For $\mathscr{X} \in \{\mathscr{U}, \mathscr{V}, \mathscr{W}\}$ denote by $k_{\mathscr{X}}$ the number of elements x_i in C that belong to $\bigcup \mathscr{X}$. Now, some of the numbers $k_{\mathscr{X}}$ may be zero, in which case the induced permutation $C_{\mathscr{X}}$ would be undefined. Nevertheless, we formally set $|C_{\mathscr{X}}| = -1$ when $k_{\mathscr{X}} = 0$. Under this convention we have $|C_{\mathscr{X}}| = k_{\mathscr{X}} - 1$ for any $\mathscr{X} \in \{\mathscr{U}, \mathscr{V}, \mathscr{W}\}$. Since $\mathscr{U} \cup \mathscr{V} = \mathscr{S}$, then $k_{\mathscr{U}} + k_{\mathscr{V}} = k + k_{\mathscr{W}}$, which implies that

$$|C_{\mathscr{U}}| + |C_{\mathscr{V}}| = |C| + |C_{\mathscr{W}}|. \tag{1}$$

Since $|C| + |C_{\mathscr{W}}| > -1$, then $|C_{\mathscr{U}}|$ and $|C_{\mathscr{V}}|$ cannot simultaneously be equal to -1. Also, if one of $|C_{\mathscr{U}}|$ or $|C_{\mathscr{V}}|$ is -1 then necessarily $|C_{\mathscr{W}}| = -1$.

Let c be the number (possibly zero) of orbits of A that permute an element of $\mathscr{U}$ and an element of $\mathscr{V}$, but fix every element of $\mathscr{W}$. Then, summing (1) over all orbits of A, we obtain

$$|A_{\mathscr{U}}| + |A_{\mathscr{V}}| = |A| + |A_{\mathscr{W}}| - c \leqslant |A| + |A_{\mathscr{W}}| = |A_{\mathscr{U} \cup \mathscr{V}}| + |A_{\mathscr{U} \cap \mathscr{V}}|. \quad ∎$$

Our next step is to define set systems with some distinguished subsets. Let $\mathscr{S}$ be a set system and let A be a permutation of $\bigcup \mathscr{S}$. Let $\mathscr{T}$ be a (possibly empty) subset of $\mathscr{S}$; we will refer to the members of $\mathscr{T}$ as *odd* sets. The triple $(\mathscr{S}, \mathscr{T}, A)$ will be called an *N-space* (N standing for Nebeský). Recall that the sets of $\mathscr{S}$ will be corner sets of the components of a relative graph $G_{A^{-1}}$. The sets in $\mathscr{T}$ will be those sets that are corners sets of components with odd Betti number.

Now let $\mathscr{U}$ be a nonempty subset of $\mathscr{S}$. Clearly, the triple $(\mathscr{U}, \mathscr{U} \cap \mathscr{T}, A_{\mathscr{U}})$ is again an N-space. It is called the *subspace* of $(\mathscr{S}, \mathscr{T}, A)$ *induced* by the subset $\mathscr{U}$.

The following is one of the central concepts of this paper. We will say that an N-space $(\mathscr{S}, \mathscr{T}, A)$ is *sparse* if

$$|A| \leqslant |\mathscr{S}| + |\mathscr{T}| - 1.$$

Actually, even a stronger version of sparseness will be needed: The N-space $(\mathscr{S}, \mathscr{T}, A)$ is said to be *uniformly sparse* if the subspace $(\mathscr{U}, \mathscr{U} \cap \mathscr{T}, A_{\mathscr{U}})$ is sparse for every nonempty subset $\mathscr{U} \subseteq \mathscr{S}$.

Let $(\mathscr{S}, \mathscr{T}, A)$ be a uniformly sparse N-space. We shall be interested in subsets that induce extremal subspaces in the following sense. A nonempty subset $\mathscr{U}$ of $\mathscr{S}$ is said to be *saturated* if it satisfies the equality $|A_{\mathscr{U}}| = |\mathscr{U}| + |\mathscr{U} \cap \mathscr{T}| - 1$.

LEMMA 6. *If $\mathscr{U}$ and $\mathscr{V}$ are intersecting saturated subsets of a uniformly sparse* N-*space then both $\mathscr{U} \cap \mathscr{V}$ and $\mathscr{U} \cup \mathscr{V}$ are saturated.*

Proof. Let $(\mathscr{S}, \mathscr{T}, A)$ be the N-space in question. For notational convenience, let $\mathscr{U}' = \mathscr{U} \cap \mathscr{T}$ and $\mathscr{V}' = \mathscr{V} \cap \mathscr{T}$. By our assumptions, we have

$$|A_{\mathscr{U}}| = |\mathscr{U}| + |\mathscr{U}'| - 1, \qquad |A_{\mathscr{V}}| = |\mathscr{V}| + |\mathscr{V}'| - 1. \tag{2}$$

The uniform sparseness of the N-space $(\mathscr{S}, \mathscr{T}, A)$ implies that the subspaces $(\mathscr{U} \cup \mathscr{V}, \mathscr{U}' \cup \mathscr{V}', A_{\mathscr{U} \cup \mathscr{V}})$ and $(\mathscr{U} \cap \mathscr{V}, \mathscr{U}' \cap \mathscr{V}', A_{\mathscr{U} \cap \mathscr{V}})$ are both sparse, that is,

$$|A_{\mathscr{U} \cup \mathscr{V}}| \leqslant |\mathscr{U} \cup \mathscr{V}| + |\mathscr{U}' \cup \mathscr{V}'| - 1 \tag{3}$$

and

$$|A_{\mathscr{U} \cap \mathscr{V}}| \leqslant |\mathscr{U} \cap \mathscr{V}| + |\mathscr{U}' \cap \mathscr{V}'| - 1. \tag{4}$$

As an easy consequence of (2)–(4) we have

$$|A_{\mathscr{U} \cup \mathscr{V}}| + |A_{\mathscr{U} \cap \mathscr{V}}| \leqslant |A_{\mathscr{U}}| + |A_{\mathscr{V}}|. \tag{5}$$

However, Lemma 5 then shows that we must have equality in (5), which in turn implies that equality must hold in both (3) and (4). Hence the sets $\mathscr{U} \cup \mathscr{V}$ and $\mathscr{U} \cap \mathscr{V}$ are both saturated. ∎

An N-space $\Omega = (\mathscr{S}, \mathscr{T}, A)$ is said to be *connected* if for any two distinct sets S and S' in $\mathscr{S}$ there exists a sequence $S = S_1, S_2, ..., S_m = S'$ in $\mathscr{S}$ such that $A(S_i) \cap S_{i+1} \neq \emptyset$ for $1 \leqslant i \leqslant m-1$. Note that Ω is connected if $|\mathscr{S}| = 1$. A subspace $\Omega_{\mathscr{U}} = (\mathscr{U}, \mathscr{U} \cap \mathscr{T}, A_{\mathscr{U}})$ of Ω is called a *component* of Ω if $\Omega_{\mathscr{U}}$ itself is connected and for every connected subspace $\Omega_{\mathscr{V}} = (\mathscr{V}, \mathscr{V} \cap \mathscr{T}, A_{\mathscr{V}})$ with $\mathscr{V} \supseteq \mathscr{U}$ we have $\mathscr{V} = \mathscr{U}$.

LEMMA 7. *Let $(\mathscr{S}, \mathscr{T}, A)$ be a uniformly sparse* N-*space and let $\mathscr{U}$ be a saturated subset of $\mathscr{S}$. Then the subspace $(\mathscr{U}, \mathscr{U} \cap \mathscr{T}, A_{\mathscr{U}})$ is connected.*

Proof. Assume that the subspace $(\mathscr{U}, \mathscr{U} \cap \mathscr{T}, A_{\mathscr{U}})$ has m components $(\mathscr{U}_i, \mathscr{U}_i \cap \mathscr{T}, A_{\mathscr{U}_i})$, $1 \leqslant i \leqslant m$, induced by a partition $\{\mathscr{U}_i\}_{1 \leqslant i \leqslant m}$ of the subset $\mathscr{U} \subseteq \mathscr{S}$. Since the original N-space is uniformly sparse, we have

$$|A_{\mathscr{U}_i}| \leqslant |\mathscr{U}_i| + |\mathscr{U}_i \cap \mathscr{T}| - 1. \tag{6}$$

By our definition of connectivity, we clearly have $|A_{\mathscr{U}}| = \sum_{i=1}^{m} |A_{\mathscr{U}_i}|$, $|\mathscr{U}| = \sum_{i=1}^{m} |\mathscr{U}_i|$, and $|\mathscr{U} \cap \mathscr{T}| = \sum_{i=1}^{m} |\mathscr{U}_i \cap \mathscr{T}|$. Summing (6) over all m components we obtain $|A_{\mathscr{U}}| \leqslant |\mathscr{U}| + |\mathscr{U} \cap \mathscr{T}| - m$. On the other hand, since $\mathscr{U}$ is saturated, we have $|A_{\mathscr{U}}| = |\mathscr{U}| + |\mathscr{U} \cap \mathscr{T}| - 1$. Comparing the last two relations yields $m = 1$; that is, the subspace $(\mathscr{U}, \mathscr{U} \cap \mathscr{T}, A_{\mathscr{U}})$ is connected. ∎

The rest of this section is devoted to proving that the property of uniform sparseness is inherited by subspaces in certain cases.

If A is a permutation of a set M not fixing an element x of M, then clearly $(x, A(x))A$ fixes x and has weight $|A| - 1$.

PROPOSITION 2. *Let $\Omega = (\mathscr{S}, \mathscr{T}, A)$ be a connected, uniformly sparse* N-*space, with $|A| \geqslant 1$. Assume that $\mathscr{T} \neq \emptyset$ and let $S \in \mathscr{T}$ be an arbitrary odd set. Then there exists an element $s \in S$ not fixed by A for which the* N-*space $\Omega_s = (\mathscr{S}, \mathscr{T} \backslash \{S\}, (s, A(s))A)$ is uniformly sparse.*

Proof. Denote by S_A the set containing all elements of S that are not fixed by A. Since Ω is connected and uniformly sparse, then evidently $S_A \neq \emptyset$. For each $s \in S_A$ let $B_s = (s, A(s))A$; thus $|B_s| = |A| - 1$. Moreover, as it follows from our definition of induced permutations, for every nonempty subset $\mathcal{U} \subseteq \mathcal{S}$ we have

$$|(B_s)_{\mathcal{U}}| \leqslant |A_{\mathcal{U}}| \quad \text{with equality if and only if } A_{\mathcal{U}}(s) = s \text{ or } S \notin \mathcal{U}. \tag{7}$$

Suppose now that the proposition is not true and that Ω_s is *not* uniformly sparse for *any* $s \in S_A$. That is, for every $s \in S_A$ we can find a nonsparse subspace $(\mathcal{U}_s, \mathcal{U}_s \cap (\mathcal{T} \setminus \{S\}), (B_s)_{\mathcal{U}_s})$ of the N-space $\Omega_s = (\mathcal{S}, \mathcal{T} \setminus \{S\}, B_s)$. We shall now investigate the properties of the collection $\{\mathcal{U}_s\}_{s \in S_A}$ of the "underlying set systems" of these nonsparse subspaces.

The fact that the subspace of Ω_s induced by $\mathcal{U}_s$ is not sparse translates into the inequality

$$|\mathcal{U}_s| + |\mathcal{U}_s \cap (\mathcal{T} \setminus \{S\})| \leqslant |(B_s)_{\mathcal{U}_s}|. \tag{8}$$

On the other hand, by our assumptions, the subspace of the original N-space Ω induced by the same $\mathcal{U}_s$ is sparse, and so

$$|A_{\mathcal{U}_s}| \leqslant |\mathcal{U}_s| + |\mathcal{U}_s \cap \mathcal{T}| - 1. \tag{9}$$

We also have the obvious inequality

$$|\mathcal{U}_s \cap \mathcal{T}| - 1 \leqslant |\mathcal{U}_s \cap (\mathcal{T} \setminus \{S\})|. \tag{10}$$

Now, combining (7) for $\mathcal{U} = \mathcal{U}_s$ with (8)–(10) we see that, in fact, equality must hold in *all* four of these inequalities. But then, the equality in (10) shows that $S \in \mathcal{U}_s$ for every $s \in S_A$. The equality in (7) then implies that $A_{\mathcal{U}_s}(s) = s$ for all $s \in S_A$. Finally, it follows from the equality in (9) that $\mathcal{U}_s$ is a saturated subset of the original N-space Ω for any $s \in S_A$.

Consider for a moment the intersection $\mathcal{V} = \mathcal{U}_s \cap \mathcal{U}_t$ for some $s, t \in S_A$. Clearly, $\mathcal{V}$ contains the set S. Thus, by Lemma 6, $\mathcal{V}$ is a saturated subset of Ω. Also, it is easy to see that $A_{\mathcal{V}}(s) = s$ and $A_{\mathcal{V}}(t) = t$. These observations are helpful in determining the properties of the intersection $\mathcal{W} = \bigcap_{s \in S_A} \mathcal{U}_s$ of *all* our sets $\mathcal{U}_s$. We see immediately that $S \in \mathcal{W}$ and $A_{\mathcal{W}}(s) = s$ for every $s \in S_A$; moreover, invoking Lemma 6 again, $\mathcal{W}$ is a saturated subset of Ω. Finally, Lemma 7 implies that the subspace $\Omega_{\mathcal{W}} = (\mathcal{W}, \mathcal{W} \cap \mathcal{T}, A_{\mathcal{W}})$ of the original N-space Ω induced by $\mathcal{W}$ is connected.

The rest of the proof is now easy. By the definition of the set S_A and the above properties of $\mathcal{W}$ we see that $A_{\mathcal{W}}(s) = s$ for every $s \in S$. In other words, the set S itself induces a connected component of the (connected) subspace $\Omega_{\mathcal{W}}$ and, hence, $\mathcal{W} = \{S\}$. But then $A_{\mathcal{W}}$ is the identity permutation on $\mathcal{W}$ and so $|A_{\mathcal{W}}| = 0$. However, since $\mathcal{W} = \{S\}$ is a saturated subset of the original N-space $\Omega = (\mathcal{S}, \mathcal{T}, A)$ and $S \in \mathcal{T}$, we have $|A_{\mathcal{W}}| = |\mathcal{W}| + |\mathcal{W} \cap \mathcal{T}| - 1 = 1 + 1 - 1 = 1$. This contradiction proves our proposition. ∎

4. A DECOMPOSITION THEOREM FOR N-SPACES

Let A be a permutation of a nonempty finite set M and set $n=|A|$. We introduce the concept of a *representative sequence* $X=(x_i)_{i=1}^n$ *for* A by means of the following inductive definition. Formally, if $n=0$, we consider the empty sequence to be the representative sequence for the identity permutation. If $n=1$, then $A=(a, A(a))$ for some $a\in M$, and a representative sequence for A is a single-element sequence $X=(x_1)$, where either $x_1=a$ or $x_1=A(a)$. For $n\geqslant 2$, a sequence $X=(x_i)_{i=1}^n$ is representative for A if $A(x_n)\neq x_n$ and the $(n-1)$-sequence $X'=(x_i)_{i=1}^{n-1}$ is representative for the permutation $A'=(x_n, A(x_n))A$. Note that this definition is possible because we have $|A'|=|A|-1$. Also, the condition $n=|A|$ automatically implies that the elements in a representative sequence must be mutually different and that x_n must be fixed by A'.

Observe that if $X=(x_i)_{i=1}^n$ is a representative sequence of A, then there exist $y_1, y_2, ..., y_n\in M$ such that

$$A=(x_n, y_n)(x_{n-1}, y_{n-1})\cdots(x_1, y_1), \tag{11}$$

where $y_n=A(x_n)$. However, it is *not* true in general that $y_i=A(x_i)$ for $i\leqslant n-1$. In order to see how the elements y_i depend on the permutation, let us introduce the concept of the *i-th partial product* A_i^X associated with (11) as

$$A_i^X=(x_i, y_i)(x_{i-1}, y_{i-1})\cdots(x_1, y_1).$$

Clearly, $A_n^X=A$ (and, formally, A_0^X is the identity permutation). Now, (11) is equivalent to

$$A=(x_n, A(x_n))(x_{n-1}, A_{n-1}^X(x_{n-1}))\cdots(x_1, A_1^X(x_1)).$$

We recall that, for $1\leqslant i\leqslant n$, the element x_i is fixed by the $(i-1)$th partial product A_{i-1}^X; in other words, $x_i\notin\{x_j, y_j : j\leqslant i-1\}$.

Let $\Omega=(\mathscr{S}, \mathscr{T}, A)$ be an N-space. Assume that for every set $S\in\mathscr{T}$ we have $|S|\geqslant 2$. A collection $\theta=\{\theta_S\colon S\in\mathscr{T}\}$ is said to be a family of *pair-separating functions for* $\mathscr{T}$ if each θ_S is a mapping that assigns to every 2-subset J of S a subset $\theta_S(J)\subseteq S$ such that $|J\cap\theta_S(J)|=1$. Let us now construct from $\mathscr{S}$ a new set system $\mathscr{S}^\theta$, called a θ-*split of* $\mathscr{S}$, in the following way. For each $S\in\mathscr{T}\subseteq\mathscr{S}$ choose a 2-subset $J_S\subseteq S$, and replace S in the set system $\mathscr{S}$ with the two sets $\theta_S(J_S)$ and $S\backslash\theta_S(J_S)$. Note that $\bigcup\mathscr{S}^\theta=\bigcup\mathscr{S}$ and $|\mathscr{S}^\theta|=|\mathscr{S}|+|\mathscr{T}|$.

Obviously, a θ-split of $\mathscr{S}$ depends heavily on both the family θ of pair-separating functions for $\mathscr{T}$ as well as the choice of the 2-subsets J_S for

$S \in \mathcal{T}$; the latter dependence does not explicitly appear in the notation but will always be clear from the context.

Let $(\mathcal{S}, \mathcal{T}, A)$ be an N-space and let θ be a family of pair-separating functions for $\mathcal{T}$. The θ-split $\mathcal{S}^\theta$ of $\mathcal{S}$ gives rise to a new N-space $(\mathcal{S}^\theta, \emptyset, A)$, which will play an important role later. For the sake of convenience, we associate with this N-space an auxiliary bipartite graph $H_\theta(A)$ defined as follows. The vertex set of $H_\theta(A)$ is the union $\mathcal{S}^\theta \cup \mathcal{C}$, where $\mathcal{C}$ is the set of all orbits of A of length $\geqslant 2$. Two vertices $S \in \mathcal{S}^\theta$ and $C \in \mathcal{C}$ are joined in $H_\theta(A)$ by an edge labeled x if $x \in S \cap C$. As an important observation, we point out that the components of the N-space $(\mathcal{S}^\theta, \emptyset, A)$ are in 1:1 correspondence with the connected components of the graph $H_\theta(A)$.

Now we are ready to state and prove our decomposition result.

THEOREM 4. *Let* $\Omega = (\mathcal{S}, \mathcal{T}, A)$ *be a connected, uniformly sparse* N-*space such that* $|S| \geqslant 2$ *for every* $S \in \mathcal{T}$. *Assume that* $\mathcal{T}$ *is endowed with a family* θ *of pair-separating functions. Then there exists a representative sequence* $X = (x_i)_{i=1}^{|A|}$ *for* A, *together with a* θ-*split* $\mathcal{S}^\theta$ *of* $\mathcal{S}$, *such that the graph* $H_\theta(A)$ *is a forest, and for* $1 \leqslant i \leqslant |A|$ *the elements* x_i *and* $A_i^X(x_i)$ *appear in different components of the* N-*space* $(\mathcal{S}^\theta, \emptyset, A_{i-1}^X)$.

Proof. Let $n = |A|$. By induction on n we first show that there exists a representative sequence X and a θ-split $\mathcal{S}^\theta$ such that the graph $H_\theta(A)$ is a forest. (Although the symbol X does not explicitly appear in the notation of the auxiliary graph, our construction of the θ-split $\mathcal{S}^\theta$ will depend on X.) The proof of the last assertion of our theorem will be postponed to the very end.

We start with noting that the connectivity of Ω implies that $|A| \geqslant |\mathcal{S}| - 1$ and that the statement is vacuously true if $n = 0$ (in which case $H_\theta(A)$ is a trivial one-vertex graph).

Suppose, first, that there exists an element $b \in \bigcup \mathcal{S}$ such that b and $A(b)$ are in different components of the N-space $\Omega' = (\mathcal{S}, \mathcal{T}, B)$, where $B = (b, A(b))A$. Then, Ω' has exactly two components, say, $\Omega_l = (\mathcal{S}_l, \mathcal{T}_l, B_l)$, where $l \in \{1, 2\}$, and $(\mathcal{S}_1, \mathcal{S}_2)$ and $(\mathcal{T}_1, \mathcal{T}_2)$ are partitions of $\mathcal{S}$ and $\mathcal{T}$, respectively. We may assume without loss of generality that $b \in \bigcup \mathcal{S}_1$ (and hence, $A(b) \in \bigcup \mathcal{S}_2$). The restriction of the family θ to $\mathcal{T}_1$ and $\mathcal{T}_2$ will be denoted by θ_1 and θ_2, respectively. Let $|B_l| = n_l$ for $l \in \{1, 2\}$. Note that $n_1 + n_2 = |B| = |A| - 1 = n - 1$. Applying the induction hypothesis to Ω_l for $l \in \{1, 2\}$, we may assume that there exists a representative sequence $X_l = (x_{l,i})_{i=1}^{n_l}$ for B_l and a θ-split $\mathcal{S}_l^{\theta_l}$ of $\mathcal{S}_l$ such that the graph $H_{\theta_l}(B_l)$ is a forest.

Let $X = (x_i)_{i=1}^n$ be a new sequence defined by: $x_i = x_{1,i}$ for $1 \leqslant i \leqslant n_1$, $x_i = x_{2,i-n_1}$ for $n_1 + 1 \leqslant i \leqslant n_1 + n_2$, and $x_n = b$. It is easy to see that X is a representative sequence for A. Indeed, recall that b is in $\bigcup \mathscr{S}_1$, $A(b)$ is in $\bigcup \mathscr{S}_2$, and elements not fixed by B_1 are fixed by B_2 and vice versa (that is, $B = B_1 B_2 = B_2 B_1$). We now define a θ-split $\mathscr{S}^\theta$ of $\mathscr{S}$ simply by setting $\mathscr{S}^\theta = \mathscr{S}_1^{\theta_1} \cup \mathscr{S}_2^{\theta_2}$. To show that the graph $H_\theta(A)$ is a forest, we assume without loss of generality that $b \in S_1 \in \mathscr{S}^{\theta_1}$ and $A(b) \in S_2 \in \mathscr{S}^{\theta_2}$. Also, we know that b is fixed by B. Now, if $A(b)$ is fixed by B as well, then $H_\theta(A)$ arises from $H_{\theta_1}(B_1) \cup H_{\theta_2}(B_2)$ by adding a new vertex corresponding to the cyclic factor $(b, A(b))$ of A and joining it to both S_1 and S_2 by new edges labeled b and $A(b)$, respectively. If $A(b)$ is not fixed by B, the symbol $A(b)$ must already appear in a nontrivial cyclic factor C of B_2. Then, $H_\theta(A)$ is obtained from $H_{\theta_1}(B_1) \cup H_{\theta_2}(B_2)$ by inserting a new edge labeled b that joins the vertex C with S_1. In both cases, the induction hypothesis implies that $H_\theta(A)$ is a forest.

Now, let the N-space $(\mathscr{S}, \mathscr{T}, (b, A(b))A)$ be connected for each $b \in \bigcup \mathscr{S}$. Then, connectivity and sparseness together quickly imply that $\mathscr{T} \neq \emptyset$. Take a set $S \in \mathscr{T}$ (recall that $|S| \geqslant 2$) and let $\mathscr{U} = \mathscr{T} \setminus \{S\}$. Let $\vartheta = \theta \setminus \{\theta_S\}$ be the family of pair-separating functions for $\mathscr{U}$. According to Proposition 2, there exists an $s \in S$ such that the N-space $\Psi = (\mathscr{S}, \mathscr{U}, (s, A(s))A)$ is uniformly sparse (and, obviously, connected). Let $B = (s, A(s))A$. Since s is not fixed by A, we have $|B| = |A| - 1 = n - 1$. We recall, however, that s is fixed by B; this fact will be important later. The induction hypothesis applied to Ψ yields the existence of a representative sequence $Y = (y_i)_{i=1}^{n-1}$ for B and a ϑ-split $\mathscr{S}^\vartheta$ of $\mathscr{S}$ such that the corresponding auxiliary graph $H_\vartheta(B)$ is a forest.

Let $X = (x_i)_{i=1}^{n}$ be the sequence obtained from Y by placing the element s at the tail of Y; formally, $x_i = y_i$ for $1 \leqslant i \leqslant n-1$ and $x_n = s$. Clearly, X is a representative sequence for A. As the next step, we establish the existence of a suitable θ-split $\mathscr{S}^\theta$ of $\mathscr{S}$. This will be done by choosing an appropriate element $z \in S$, $z \neq s$, and making use of the pair-separation function θ_S, applied to the 2-subset $\{s, z\}$.

Let $S' \in \mathscr{S}^\vartheta$ be the set that contains the element $A(s)$. If $S' = S$ then we choose $z = A(s)$, noting that $A(s) \neq s$. If S and S' are in different components of the graph $H_\vartheta(B)$, then we choose for z an arbitrary element in S that is distinct from s. Finally, if $S' \neq S$ and both S and S' are in the same connected component of $H_\vartheta(B)$, then there is a unique path L in $H_\vartheta(B)$ from S to S'. Let t be the label of the edge of L incident with the vertex S. It follows that the symbol $t \in S$ also appears in a nontrivial cyclic factor of the permutation B, and since s is fixed by B, we have $t \neq s$. In this case we choose $z = t$.

Let $J = \{s, z\}$. We construct a θ-split $\mathscr{S}^\theta$ of $\mathscr{S}$ from the ϑ-split $\mathscr{S}^\vartheta$ by removing the set S from $\mathscr{S}^\vartheta$ and adding to $\mathscr{S}^\vartheta$ the two sets $S_1 = \theta_S(J)$ and $S_2 = S \setminus S_1$. (Informally, we obtain $\mathscr{S}^\theta$ from $\mathscr{S}^\vartheta$ by "splitting" the set S.) Let $H_\theta(B)$ be the bipartite graph associated with the N-space $(\mathscr{S}^\theta, \emptyset, B)$.

Clearly, the graph $H_\vartheta(B)$ can be obtained back from our new graph $H_\theta(B)$ by identifying the vertices S_1 and S_2 into S. Consequently, $H_\theta(B)$ is again a forest and, moreover, S_1 and S_2 are in different components of $H_\theta(B)$ (otherwise we would obtain a cycle in $H_\vartheta(B)$ after identifying S_1 and S_2, contrary to our induction hypothesis).

It remains to show that the auxiliary graph $H_\theta(A)$ is a forest; this is best done by explaining how $H_\theta(A)$ arises from $H_\theta(B)$. The key fact is to observe that our construction of the θ-split $\mathscr{S}^\theta$ (that is, the way the element z and the sets S_1 and S_2 have been chosen) guarantees that the elements s and $A(s)$ belong to sets in $\mathscr{S}^\theta$ that are in *different* components of the graph $H_\theta(B)$. Now, $H_\theta(A)$ is obtained from $H_\theta(B)$ by adding and edge or a path of length two (according to whether B moves or fixes the element $A(s)$) that connects the components in question; details are the same as in the first part of the proof. Thus, $H_\theta(A)$ is a forest, as claimed, and this completes the induction step.

To finish the proof we have to make sure that, for $1 \leqslant i \leqslant n$, the elements $x_i \in X$ and $A_i^X(x_i)$ are in different components of the N-space $(\mathscr{S}^\theta, \varnothing, A_{i-1}^X)$. Consider the corresponding auxiliary graphs $H_\theta(A_{i-1}^X)$. The above discussion shows that, starting with the forest $H_\theta(A) = H_\theta(A_n^X)$, for $i = n, n-1, ..., 1$ the graph $H_\theta(A_{i-1}^X)$ arises from $H_\theta(A_i^X)$ by removing either an edge or a path of length two, leaving the sets containing x_i and $A_i^X(x_i)$ in *different* components of $H_\theta(A_{i-1}^X)$. It is now sufficient to realize that the connected components of $H_\theta(A_{i-1}^X)$ are in 1:1 correspondence with the components of the N-space $(\mathscr{S}^\theta, \varnothing, A_{i-1}^X)$. ∎

It is easy to check that the graph $H_\theta(A)$ (and hence, the N-space $(\mathscr{S}^\theta, \varnothing, A)$) has exactly $|\mathscr{S}| + |\mathscr{T}| - |A|$ components.

5. PROOF OF THE MAIN RESULT

For the purpose of induction, we prove a slightly stronger result than the one stated in Theorem 3. Recall that, by Proposition 1, we are required to show the *existence* of an embedding of G with $y_G + 1$ outer faces.

PROPOSITION 3. *Every connected relative graph G has an embedding with $y_G + 1$ outer faces. Moreover, if G is prime, then for any pair of corners a and b there exists an embedding of G with two outer faces such that a and b do not belong to the same outer face.*

Proof. Let m_G be the number of corners of the relative graph $G = (M, \Pi, Q)$. We use induction on the number $2m_G + y_G$. The theorem is obviously true for the (unique) relative graph with just one corner. In what follows we assume that G is a connected relative graph with at least two corners.

Let A be a critical permutation for G. We consider two cases, according to whether or not the relative graph G_A is connected.

Case 1. *The relative graph G_A is connected.* Suppose $Q = \mathsf{id}$. Then G must be a relative graph with just one vertex and a number of inner loop-faces. In this case, clearly we have $y_G = 0$ and G has an embedding with $y_G + 1 = 1$ outer face. Henceforth, we assume $Q \neq \mathsf{id}$ and that a is a corner of G such that $b = Q(a) \neq a$.

Let $H = G_{(a,b)}$. Since A is critical, applying the transposition (a, b) to G cannot yield more components than $c_G(A) = 1$, and hence, H is necessarily connected. Also, if B is an extremal permutation for H and we write $|B(a,b)| = |B| + \delta$, where $\delta \in \{-1, 1\}$. Then

$$y_H = c_H(B) + o_H(B) - |B| - 1 = y_G(B(a,b)) + \delta \leqslant y_G + \delta. \tag{12}$$

The connectivity of G_A also allows us to estimate the weight of the critical permutation A:

$$y_G = c_G(A) + o_G(A) - |A| - 1 \leqslant 1 - |A|, \quad \text{so } y_G + |A| \leqslant 1.$$

It follows that either $y_G = 1$ and $A = \mathsf{id}$, or $y_G = 0$ and $|A| \leqslant 1$. We handle these two possibilities separately.

Subcase 1.1. $y_G = 1$ and $A = \mathsf{id}$. Since $G = G_A$, then by Lemma 9, G must be prime. If $y_H \geqslant 2$ then equality must hold in (12) and, moreover, $\delta = 1$. Consequently, $y_G(B(a,b)) = 1$ and therefore $B(a,b)$ would be an extremal permutation for G of weight at least $\delta = 1$, contrary to the fact that G is prime. Hence, we conclude that $y_H = 0$.

Now, since $m_H = m_G$ and $2m_H + y_H < 2m_G + y_G$, we may apply induction hypothesis to H and assume the existence of an embedding of H with a single outer face. Since $G = G_{(a,b)(a,b)} = H_{(a,b)}$, then by Lemma 6, G can be embedded with two outer faces neither of which contains both a and b.

Subcase 1.2. $y_G = 0$ and $|A| \leqslant 1$. Since $y_H \geqslant 0$ then (12) implies $y_H = \delta = 1$. We claim that B must be trivial and hence, since B is arbitrary, H is prime. By way of contradiction, suppose that $|B| \geqslant 1$. It follows that $|B(a,b)| \geqslant 2$. The fact that A is critical implies that $c_G(B(a,b)) = 1$. But then

$$0 = y_G \leqslant c_G(B(a,b)) + o_G(B(a,b)) - |B(a,b)| - 1 \leqslant 1 - |B(a,b)|,$$

a contradiction. Consequently, B is trivial and H is prime.

Now since $Q(a)=b$ then a constitutes a loop-face in $H=G_{(a,b)}$. By Lemma 4, the (connected) relative graph $K=H-a$ is prime; in particular, $y_K=y_H=1$ and the vertex v containing the corner a in H must contain at least two other corners. Let c be a corner of v other than a or b. Since $m_K=m_H-1=m_G-1$, we have $2m_K+y_K<2m_G+y_G$. We therefore may use induction hypothesis for K; as the result we have an embedding of K with two outer faces such that b and c belong to different outer faces. We now attach the loop-face comprising corner a to this embedding at the vertex v inside the outer face that contains c, thus obtaining an embedding of H with two outer faces neither of which contains both a and b. Finally, we apply the transposition (a, b) to H; the result is an embedding of G with a single outer face (by Lemma 1).

Case 2. *The relative graph* G_A *is disconnected.* Note first that G cannot be prime. Let $H_i=(M_i, \Pi_i, Q_i)$, $1\leqslant i\leqslant m$, be the components of G_A. By Lemma 2, we have $y_{H_i}\in\{0, 1\}$, and therefore we choose this labeling so that $y_{H_i}=1$ for $1\leqslant i\leqslant k$ and $y_{H_i}=0$ for $k+1\leqslant i\leqslant m$. Lemma 3 then implies that the relative graphs H_i, $1\leqslant i\leqslant k$ are prime. Clearly for $1\leqslant i\leqslant m$ we have $2m_{H_i}+y_{H_i}<2m_G+y_G$, and we may thus apply the induction hypothesis to each component H_i of G_A. We therefore assume the existence of an embedding P_i of H_i, $k+1\leqslant i\leqslant m$ with a single outer face. Also, for $1\leqslant i\leqslant k$ and each pair of corners a, b of H_i, we may assume the existence of an embedding $P_i[a, b]$ of H_i with exactly two outer faces, neither of which contains both a and b.

Let $\mathscr{S}=\{M_i\colon 1\leqslant i\leqslant m\}$ and let $\mathscr{T}=\{M_i\colon 1\leqslant i\leqslant k\}$. Further, for each 2-subset $J=\{a, b\}$ of $M_i\in\mathscr{T}$ let $\theta_{M_i}(J)$ be the set of corners of one of the faces of the embedding $P_i[a, b]$. Clearly, the set $\theta=\{\theta_{M_i}\colon M_i\in\mathscr{T}\}$ is a family of pair-separating functions for $\mathscr{T}$, as defined in Section 2.

Let now $B=A^{-1}$ and consider the N-space $\Omega=(\mathscr{S}, \mathscr{T}, B)$. Recalling the way how connected N-spaces have been introduced in Section 2, it is easy to see that the connectivity of G implies that the N-space Ω is connected as well. Our next aim is to show that Ω is uniformly sparse, that is, for each induced N-space $\Omega_{\mathscr{U}}=(\mathscr{U}, \mathscr{U}\cap\mathscr{T}, B_{\mathscr{U}})$, where $\varnothing\neq\mathscr{U}\subseteq\mathscr{S}$ we have

$$|B_{\mathscr{U}}|\leqslant|\mathscr{U}|+|\mathscr{U}\cap\mathscr{T}|-1. \tag{13}$$

Let $D=B_{\mathscr{U}}B^{-1}=B_{\mathscr{U}}A$. It is a matter of routine to see that the way an induced permutation is defined implies that

$$|D|=|A|-|B_{\mathscr{U}}|. \tag{14}$$

Now, let us estimate the number of components of the relative graph G_D. Since G_D arises from G_A by applying the permutation $B_{\mathscr{U}}$, it follows that every component of G_A not containing a corner of $\bigcup\mathscr{U}$ is also a component of G_D. In other words, $c_G(D) \geqslant m - |\mathscr{U}| + 1$. Similarly, it is easy to see that $o_G(D) \geqslant k - |\mathscr{U} \cap \mathscr{T}|$. Therefore,

$$y_G \geqslant y_G(D) \geqslant m - |\mathscr{U}| + 1 + k - |\mathscr{U} \cap \mathscr{T}| - |D| - 1. \tag{15}$$

The fact that A is critical for G means that $y_G = m + k - |A| - 1$; combining this with (14) and (15) yields (13), that is, our N-space Ω is uniformly sparse.

We now may apply Theorem 4 to the N-space $\Omega = (\mathscr{S}, \mathscr{T}, B)$ with the family θ of pair-separating functions for $\mathscr{T}$. As the result, there exists a representative sequence $X = (x_i)_{i=1}^n$, $n = |B|$ for the permutation B, together with a θ-split $\mathscr{S}^\theta$, such that for $1 \leqslant i \leqslant n$ the corners x_i and $B_i^X(x_i)$ are in different components of the N-space $\Omega_{i-1} = (\mathscr{S}^\theta, \varnothing, B_{i-1}^X)$. Let us see how this translates into the language of embeddings.

Let $\mathscr{S}^\theta = \{M_1^{a_1}, M_1^{b_1}, M_2^{a_2}, M_2^{b_2}, ..., M_k^{a_k}, M_k^{b_k}, M_{k+1}, ..., M_m\}$ be our θ-split of $\mathscr{S}$, where each set $M_i \in \mathscr{T}$ of corners of H_i, $1 \leqslant i \leqslant k$, has been partitioned into two sets $M_i^{a_i}$ and $M_i^{b_i}$, according to a suitable 2-subset $J = \{a_i, b_i\}$ (that is, in the above notation, $\theta_{M_i}(J)$ is either $M_i^{a_i}$ or $M_i^{b_i}$). But here each such partition is induced by a 2-outer-face embedding $P_i^* = P_i[a_i, b_i]$ of H_i, $1 \leqslant i \leqslant k$. For the sake of completeness, for $k+1 \leqslant i \leqslant m$ let P_i^* be an arbitrary embedding of H_i with a single outer face. Note that the components of the N-space $\Omega_0 = (\mathscr{S}^\theta, \varnothing, \mathsf{id})$ are in 1:1 correspondence with the sets of corners of the total of $m+k$ outer faces in the embeddings P_i^*, $1 \leqslant i \leqslant m$. Thus, we can interpret the "x_1"-part of Theorem 4 as follows: There exists an embedding $\psi_0 = \bigcup_{i=1}^m P_i^*$ of the relative graph $G_A = \bigcup_{i=1}^m H_i$ such that the corners x_1 and $y_1 = B_1^X(x_1)$ appear in different outer faces of ψ_0, say, M and M'.

Consider now the embedding ψ_1 of the relative graph G_{A_1} for $A_1 = (x_1, y_1) A$, obtained from the embedding ψ_0 of G_A by applying the transposition (x_1, y_1), which results in pasting together the faces M and M' into a single outer face. Observe that the components of G_{A_1} again are in 1:1 correspondence with components of the N-space $\Omega_1 = (\mathscr{S}^\theta, \varnothing, (x_1, y_1))$. By the "topological translation" of Theorem 4, the elements x_2 and $y_2 = B_2^X(x_2)$ are in different faces of the embedding ψ_1. So we may form a new embedding ψ_2 of the relative graph G_{A_2} for $A_2 = (x_2, y_2)(x_1, y_1) A$ which identifies the two faces, and whose components are in 1:1 correspondence with components of the N-space $\Omega_2 = (\mathscr{S}^\theta, \varnothing, (x_2, y_2)(x_1, y_1))$, etc. It is clear that, iterating this process $n = |B|$ times, we finally end up with an embedding ψ_n of the relative graph G_{A_n} for $A_n = (x_n, y_n) \cdots (x_1, y_1) A$, $y_i = B_i^X(x_i)$, with $(m+k) - n$ outer faces. But since $X = (x_i)_{i=1}^n$ was a representative sequence for $B = A^{-1}$, we have $A_n = BA = \mathsf{id}$ and, hence $G_{A_n} = G$. Also, since $y_G = m + k - n - 1$, the number of outer faces of the embedding ψ_n of G is $m + k - n = y_G + 1$, as claimed. This completes the proof of Theorem 3. ∎

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